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ON SOME HYDRODYNAMIC MODELS OF DRAINAGE

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A single drain model is used for investigating with due regard to physical parameters the local depression appearing over a drain in a stratum of unbounded depth and extension under certain conditions, one of which is the increase of the drain discharge rate. Such increase is shown to be limited to some value (for given width of the section free of flooding and depth of the drain sink) that corresponds to the limit case [1] in which the depression curve has a cusp.

The condition of local depression existence in the case of more general models of plane zero-pressure head flows were earlier formulated by the author in [2 — 4] in terms of conformal mapping parameters, but the physical aspects of this effect were only tentatively outlined there.

The continuation of solution for the above model with respect to one of the mapping parameters leads to the appearance of new hydrodynamic models such as the flow to a point sink on the surface of a vertical screen, the flow over the screen, and over an underground projection. In conclusion a model is considered which unifies the preceding ones using the flow over a screen with partial interception by a drainage sink in the screen surface, as the basic scheme. In cases close to the limit simple approximate relations are established between seepage properties of the stream and controlling geometrical parameters. Results of numerical calculations are presented in the form of curves.

1. Model 1. Let us consider a plane zero-pressure head steady seepage in an unbounded (with respect to depth and extension) stratum from the ground surface flooded everywhere except a band of width $2l$, to a tubular drain laid in the middle of it. The right-hand half of the seepage region bounded in this case by the depression curve 1 is schematically shown in Fig. 1, a, where the drain is simulated by the point sink D whose discharge rate we denote by $2Q$. We shall use the following dimensionless quantities: the complex coordinate $z = x + iy$ and the complex potential $\omega = \varphi + i\psi$ related to the respective actual quantities z_f and ω_f by formulas ($\varkappa$ is the seepage coefficient)

$$z = z_f / l, \quad \omega = \omega_f/(\varkappa l) \tag{1.1}$$

Region ω is shown in Fig. 1, b.

When $Q = 0$ the free surface AB of the ground water coincides with the plane $y = 0$ and the fluid pressure p increases with the depth of stratum in conformity with the hydrostatic law $p = \gamma y$, where γ is the specific gravity of water. If we assume that when the sink is in operation the pressure at every fixed point of region z, except the sink itself, is continuously dependent on the rate of flow, then, taking the aforesaid into account, we can conclude that, at least for low flow rates, the pressure over some initial part of section AD must increase as before, and then drop down to $p = -\infty$ at point D. There is then a slit along A in the Joukowskii function plane $\theta = \omega + iz$, denoted by the numeral 1 in Fig. 1, c. The slit top F corresponds to the point of maximum pressure in section AD.

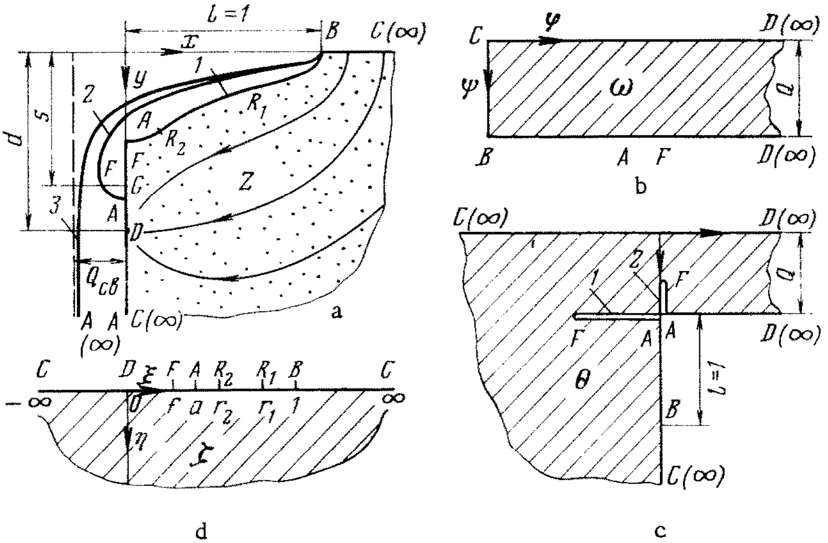

Fig. 1

The mapping of regions ω and θ onto the half-plane $\operatorname{Im} \zeta \geqslant 0$ (Fig. 1, d) yields

$$\omega = \frac{Q}{\pi} \int_{\zeta}^{1} \frac{d\zeta}{\zeta\sqrt{1-\zeta}} + iQ = q \operatorname{arth} \sqrt{1-\zeta} + iQ, \quad q = \frac{2}{\pi} Q \tag{1.2}$$

$$\theta = i\frac{Q}{\pi}\frac{\sqrt{a}}{f} \int_{a}^{\zeta} \frac{(\zeta - f)\, d\zeta}{\zeta\sqrt{\zeta - a}} + iQ = \tag{1.3}$$

$$iq\left(\frac{a}{f}\sqrt{\frac{\zeta}{a} - 1} - \operatorname{arctg}\sqrt{\frac{\zeta}{a} - 1}\right) + iQ$$

$$z = i(\omega - \theta) = \frac{q}{2}\left[\frac{\sqrt{a}}{f} \int_{a}^{\zeta} \frac{(\zeta - f)\, d\zeta}{\zeta\sqrt{\zeta - a}} + i\int_{\zeta}^{1} \frac{d\zeta}{\zeta\sqrt{1-\zeta}}\right] = \tag{1.4}$$

$$q\left[\left(\frac{a}{f}\sqrt{\frac{\zeta}{a} - 1} - \operatorname{arctg}\sqrt{\frac{\zeta}{a} - 1}\right) + i \operatorname{arth}\sqrt{1-\zeta}\right]$$

Using (1.4) we obtain that along section AC $(\zeta < a)$

$$y = q\left(\frac{a}{f}\sqrt{1 - \frac{\zeta}{a}} - \ln\frac{1 + \sqrt{1-\zeta}}{\sqrt{a} + \sqrt{a - \zeta}}\right) \tag{1.5}$$

By setting in (1.4) $\zeta = 1,\ z = 1$ and in (1.5) $\zeta = 0,\ y = d$ we specify the width of the band free of flooding and the drainage sink depth. As the result we obtain for parameters a and f the system of equations

$$q(\alpha\lambda - \operatorname{arctg}\alpha) = 1 \qquad (1.6)$$

$$q(\lambda + \ln\sqrt{1+\alpha^2}) = d \qquad \left(\alpha = \sqrt{\frac{1}{a} - 1}, \quad \lambda = \frac{a}{f}\right) \qquad (1.7)$$

Let us investigate the dependence of the flow pattern on the sink discharge rate with fixed geometrical parameters, starting with the unperturbed state $q = 0$ for which with allowance for (1.7) we have

$$z = \sqrt{\frac{\zeta - a}{1-a}}, \quad \omega = 0, \quad \theta = iz; \quad a = a_0 = \frac{d^2}{1+d^2} \qquad (1.8)$$
$$f = f_0 = 0, \quad \lambda = \lambda_0 = \infty$$

Then using system (1.6), (1.7) and denoting differentiation with respect to q by a prime, we obtain

$$f = a\frac{\alpha d - 1}{\ln\sqrt{1+\alpha^2} + d\operatorname{arctg}\alpha}, \quad q = \frac{\alpha d - 1}{\alpha\ln\sqrt{1+\alpha^2} + \operatorname{arctg}\alpha} \qquad (1.9)$$
$$\lambda = \frac{d}{q} - \ln\sqrt{1+\alpha^2}$$

$$a' = -\frac{2a^2\alpha(\alpha\cdot\ln\sqrt{1+\alpha^2} + \operatorname{arctg}\alpha)}{q(\lambda - 1)}, \quad f' = \frac{a'}{2\lambda} + \frac{f^2}{q^2}(\alpha + d) \qquad (1.10)$$

On the strength of the last of equalities (1.8) and the continuous dependence of λ on q we have $\lambda > 1$ in some finite interval of the drain flow rate increase (from the initial value $q = 0$). According to (1.10) $a' < 0$ and, consequently, for any q in the considered interval parameters a and f are uniquely determined by system (1.6), (1.7), or by the equivalent to it system of the first two of Eqs. (1.9). It also follows from (1.9) and (1.10) that λ monotonically decreases as q increases, and for some $q = q_*$ we obtain $\lambda = 1$, i.e. $a = f = a_*$, with a_* and q_* determined, respectively, from the first and second of formulas (1.9) when $a = f$. Hence, when $q = q_*$, point F coincides with point A of the depression curve, while along AD, using (1.3) we obtain for pressure the following relationship

$$p/\gamma = -q_*(\operatorname{arth}\sqrt{1 - \zeta/a_*} - \sqrt{1 - \zeta/a_*}) < 0, \quad 0 < \zeta < a_* \qquad (1.11)$$

For the depression curve AB, where $d\omega = dy$; we have in conformity with (1.2) – (1.4)

$$\frac{dy}{dx} = i\frac{d\omega}{d\theta} = -\frac{f}{\sqrt{a}}\frac{\sqrt{\zeta - a}}{(\zeta - f)\sqrt{1-\zeta}} \qquad (1.12)$$

$$\frac{d^2y}{dx^2} = -\frac{f^2}{qa}\frac{\zeta P(\zeta)}{(\zeta - f)^3(1-\zeta)^{3/2}}$$

$$P(\zeta) = 2\zeta^2 - (1 + 3a)\zeta + [(a - f) + a(1 + f)] = \qquad (1.13)$$
$$2(\zeta - r_1)(\zeta - r_2)$$
$$r_{1,2} = a + (1 - a \pm \sqrt{D})/4, \quad D = (1 - a)D_1$$
$$D_1 = 1 + 8f - 9a$$

It can be readily checked that when $D > 0$ we have $r_{1,2} \in (a, 1)$. Turning to (1.12) we note that in this case the curve AB has two inflection points $R_1(r_1)$ and $R_2(r_2)$ (Fig. 1, a) bounding on it a convexity section, and on sections AR_2 and R_1B the compression curve is concave. When $D < 0$ the depression curve is concave throughout its length.

In accordance with (1.13) $\operatorname{sign} D = \operatorname{sign} D_1$, and because of (1.10) the expression for D_1 monotonically increases with the increase of q and is evidently positive for $q \approx q_* \ (a \approx f)$. On the other hand, for low q we have $D_1 \approx 1 - 9a_0$ (see (1.8)). Owing to this we have two alternatives:

a) $a_0 < 1/_9 \ (d < \sqrt{1/_8})$; we have $D > 0$ when $0 \leqslant q \leqslant q_*$;

b) $a_0 > 1/_9 \ (d > \sqrt{1/_8})$; in this case $D < 0$ when $0 \leqslant q < q_r$, and $D > 0$ when $q_r < q \leqslant q_*$. The quantity q_r is determined by equality (1.9) with $f = (1 - 9a)/8 \ (D_1 = D = 0)$.

From the physical point of view the above means that when the width $2l$ of the drained (not flooded) section of the ground surface exceeds

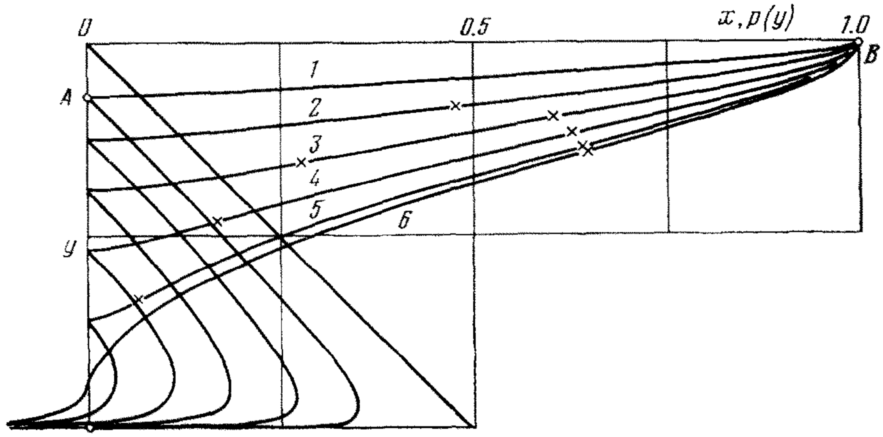

Fig. 2

the depth d of the drainage sink by more than $4\sqrt{2}$ times, the depression curve has two inflection points R_1 and R_2 throughout the interval $(0, q_*)$ of values of q. In the opposite case it remains concave as long as $q < q_r$, and monotonically flattens out with increasing distance from the flooded surface. With the subsequent increase of q on AB a section of local depression AR_1 appears in the drain zone. As q approaches q_* we have in accordance with (1.13) $r_2 \to a$, and the section AR_2 of the depression curve flattens out within the aforementioned section AR_1. It follows from (1.12) that the tangent to the depression curve is horizontal at point A for all $q < q_* \ (a > f)$ [1], and the ordinate of that point $y_A = q \operatorname{arth} \sqrt{1-a}$ monotonically increases with increasing q, as implied by (1.10). At the limit with $q = q_*$, $a = f$ from (1.13) and (1.12) we have $r_2 = a$, $(dy/dx)_{\zeta=a} = -\infty$ and, consequently, point R_2 coincides with point A which becomes the cusp of the depression curve.

Thus the local depression appears already at a very early stage of draining as the extension of the depression curve in relation to the drain depth is increased to a specific value, while when the extension is small, it appears at a latter stage of draining and develops with the subsequent increase of the drain flow rate to some admissible limit value. We call critical the mode of flow which takes place under these conditions. The necessary condition for its realization is according to (1.11) the maintenance of pressure below atmospheric along the whole section AD [1].

The depression curves shown in Fig. 2 were calculated for $d = 0.5$ $(a_0 = 0.2)$ and $Q = 0.0702, 0{,}1173, 0.1670, 0.2101, 0.2469, 0.2600$ (curves *1—6*, respectively). The inflection points are indicated by crosses. Local depression appears in this case when $Q = Q_r = 0.1173$ at point $R_{1,2}$ $(0.4771, 0.0837)$; ; using (1.4) and (1.13) it is possible to show that generally $x(R_{1,2}) < 0.5$ at the instant of appearance of inflection points on the depression curve. With increasing Q the local depression widens and deepens. in the critical mode when $Q = Q_* = 0.2600$, it is bounded by point R_1 $(0.6484, 0.1374)$ with $a = f = a_* = 0.01756$, $y(R_2, A) = y_* = 0.4485$. For each calculated Q curves of pressure variation along AD were also determined. A set of such curves may be used for determining Q for a specified pressure head at some point of the stream within section AD.

Let us consider two cases close to limiting ones.

1) $d \approx 0$ which in accordance with (1.1) relates either to the lifting of the sink D or widening of the drained section. In this scase, on the strength of (1.7) — (1.9) we have

$$a \leqslant a_0 \approx (d/l)^2, \quad q \approx d \,/\, \ln \sqrt{1/a} = O\,[d/\ln\,(l/d)]$$

expressed in terms of actual quantities l_f, d_f, and q_f, but with the subscript f omitted.

2) $d \approx \infty$. We write the equation for a_* in the form

$$\sqrt{\frac{1}{a_*} - 1} - \operatorname{arctg} \sqrt{\frac{1}{a_*} - 1} = \frac{1}{d}\left(1 + \ln \sqrt{\frac{1}{a_*}}\right) \qquad (1.14)$$

We immediately establish that $da_* \,/\, dd > 0$, hence a_* is bounded below, as d is increased. Then, turning to (1.14), we conclude that $a_* \to 1$ as $d \to \infty$ and, since $a > a_*$, using actual quantities, we obtain

$$1 - a_* \approx (3l/d)^{2/3}, \quad 1 - a = O\,[(3l/d)^{2/3}] \qquad (1.15)$$

$$y_{AB} = q \operatorname{arth} \sqrt{1 - \zeta} = q \cdot O\,[(3l/d)^{1/3}], \quad q = \frac{d}{\lambda + \ln \sqrt{1 + \alpha^2}} \approx \frac{d}{\lambda}, \quad q_* \approx d$$

Which implies that in the case of limited q the depth of sink as well as the reduction of l is accompanied by a rise of the depression curve, although on the other hand, the increase of d makes possible the increase of drainage rate. As the result, for the critical mode we have $y_A{}^* \approx (3ld^2)^{1/3}$, hence the sinking of a drain widens the range of possible lowering of the ground water level.

2. Model 2. The one-to-one correspondence between the rate of flow Q and parameter a in model 1 is defined by the first of equalities (1.10) and provides the possibility of representing the flow pattern, investigated in Sect. 1 and defined by formulas (1.2) – (1.4) for $\lambda > 1$, as the result of decreasing parameter a from a_0 to a_*. However (1.2) – (1.4) may be considered also for $a \in (0, a_*)$. Let us clarify the result of such extension of the solution.

Using (1.9) and (1.10) we obtain

$$\frac{d\lambda}{da} = \frac{1}{2a}\left[1 + \frac{(\lambda - 1)\,d}{a\alpha\,(\alpha d - 1)}\right] \tag{2.1}$$

i.e. $d\lambda / da > 0$ when $a = a_*$ $(\lambda = 1)$ and, consequently, $\lambda < 1$ in the finite interval (a_1, a_*) of $a < a_*$. If then $a_1 > 0$ and $\lambda(a_1) = 1$, by virtue of (2.1) we have $d\lambda / da > 0$ when $a = a_1$ and $\lambda > 1$ in some interval $(a_1, a_1 + \delta)$. The contradiction with the previous statement means that the relation $\lambda < 1$ $(a < f)$ is maintained throughout the interval $(0, a_*)$ of parameter a. Taking this into account, in conformity with (1.10), we conclude that

$$dq / da > 0, \quad df / da > 0; \quad a \in (0, a_*) \tag{2.2}$$

It follows from (1.4) that the abscissa x of the depression curve AB (which together with the slit along it in the θ -plane are denoted in Fig. 1, c by the numeral 1) now decreases on section AF, so that $x < 0$ when $a < \zeta < r_0$; r_0 is determined using the equation

$$u = \operatorname{tg}(\lambda u), \quad u = \sqrt{r_0 / a - 1} \tag{2.3}$$

The left-hand half of the depression curve, which is symmetric to the right-hand one, similarly enters the half-plane $x > 0$. The resulting two-sheeted superposition in the seepage plane makes the previous treatment of solution unacceptable, however, when $a < f$, it is possible to give it a physical interpretation in the scheme of one-sided inflow. For this we shall in addition to the above exposition, analyze the flow pattern along section AC, where by virtue of (1.2) – (1.4) the relations

$$\frac{dy}{d\zeta} = -\frac{q\tau(\zeta)}{2\zeta\sqrt{1-\zeta}}, \quad \tau(\zeta) = \frac{1}{w_y} = \frac{dy}{d\omega} = \tag{2.4}$$

$$1 - \frac{\sqrt{a}}{f}\frac{(f-\zeta)\sqrt{1-\zeta}}{\sqrt{a-\zeta}}; \quad -\infty < \zeta < a$$

$$\lim_{\zeta\to a-0,\ \zeta\to-\infty} \tau(\zeta) = -\infty, \quad \tau(0) = 0$$

$$\frac{d\tau}{d\zeta} = \frac{\sqrt{a}}{2f} P(\zeta)\left[(1-\zeta)(a-\zeta)^3\right]^{-1/2}$$

are satisfied (w_y is the vertical component of the seepage rate).

Reverting to (1.13) we observe that in the case of $a < f$, when $r_2 \in (-\infty, a)$, the point $R_2(r_2)$ represents the maximum of function $\tau(\zeta)$ along section AC. Since $r_2 \approx a$, $P(0) \approx 1 + f > 0$ when $a \approx f$, hence directly after parameter a had passed through the value a_* followed by its further decrease, we have in some interval $r_2 \in (0, a)$, $R_2 \in AD$. In accordance with (2.4) we have $\tau(\zeta) > 0$ in the interval $(0, g)$; the quantity g for which $\tau = 0$, $w_y = \infty$ is determined by the equality

$$g = (^1/_2)[1 + 2f - \sqrt{1 + 4(f^2/a)(1-\lambda)}] \tag{2.5}$$

Thus in some finite interval (a_{**}, a_*) of values of parameter a the ordinate y along section AC first decreases, reaches its minimum at point $G(g)$, and then increases up to point C where $y = \infty$, and $D \in GC$. Hence in the z-plane we have a vertical screen with its top at point G; a part of the stream flows over it, and in reverse motion reaches the sink D on the inner surface of the screen. The ordinate s of point G is defined by formula (1.5) with $\zeta = g$

$$s = q\left(\lambda\sqrt{1 - \frac{g}{a}} - \ln\frac{1 + \sqrt{1-g}}{\sqrt{a} + \sqrt{a-g}}\right) \tag{2.6}$$

To clarify the pattern further we investigate the asymptotics of solution when $a \approx 0$ $(\alpha \approx \infty)$. Using (1.9) we obtain

$$q \approx \frac{d}{\ln\sqrt{1/a}}, \quad f \approx \sqrt{a}\,\frac{d}{\pi d/2 + \ln\sqrt{1/a}} \approx \sqrt{a}\,\frac{q}{1+Q} \tag{2.7}$$
$$\lambda \approx \sqrt{a}\,\frac{1+Q}{q}$$

from which in conformity with (2.5) and (2.6) we obtain

$$g \approx -\frac{f^2}{a} \approx -\left(\frac{q}{1+Q}\right)^2 \tag{2.8}$$
$$s \approx q\left(1 + \ln\frac{2}{\sqrt{-g}}\right) \approx q\left[1 + \ln\left(\frac{2}{q} + \pi\right)\right]$$

Using formulas (2.3), (2.7), (2.8), and (1.4), for the ordinate y_0 of point $R_0(r_0)$ lying on the depression curve above point G we obtain

$$\frac{r_0}{a} > \frac{f}{a} \to \infty, \quad \lambda u \approx \frac{\sqrt{ar_0}}{f} \approx \frac{\pi}{2}, \quad \sqrt{r_0} \approx \frac{\pi}{2}\sqrt{-g} \approx 1 \tag{2.9}$$
$$y_0 = q\,\mathrm{arth}\sqrt{1 - r_0} \approx q\ln\frac{2}{\sqrt{r_0}} \approx q\ln\left(\frac{4}{\pi q} + 2\right) \approx s - 1.4516q$$

Then, in accordance with (1.4) and (2.7) we can write

$$y_A = q\,\mathrm{arth}\sqrt{1-a} \approx q\ln\frac{2}{\sqrt{a}} \approx d + q\ln 2 \tag{2.10}$$
$$x_F = q(\lambda\sqrt{f/a - 1} - \mathrm{arctg}\sqrt{f/a - 1}) \approx -Q$$

The first relation in (2.8) implies that when parameter a is decreased the sink D $(\zeta = 0)$ appears on the screen external surface. From (2.7) and (2.8) also follows that $q \to 0$ as $a \to 0$, the screen moves upward screening the sink more and more from the catchment contour, i.e. the flooded part of the ground surface.

Depression curves calculated by Eq. (1.4) are shown in Fig. 3 by the solid lines $1-5$ for the previously specified sink depth $d = 0.5$ and $a = 3{,}7836 \cdot 10^{-3}, 10^{-4}$, 10^{-6}, 10^{-12}, and 10^{-30} for $Q = 0.2390$, 0.1666, 0.1134, 0.0569, and 0.0227. The respective values of s are: $s = 0.5$, 0.422, 0.319, 0.183, and 0.086, which are shown by oblique segments that connect one or the other curve with the screen top. The meaning of dash lines will be explained in Sect. 3.

The calculations show that, as parameter a is decreased from $a_* = 0.01756$ to $a_{**} = 0.003784$, parameter s monotonically increases from 0.4485 to 0.5, which means that the screen sinks and for $a = a_{**}$ the sink D is at its top. In the interval $[0, a_{**}]$ of the above calculation values of parameter a, the quantity s monotonically decreases from 0.5 to zero, as a is decreased. Taking this into consideration it is possible to invert function $s(a)$ and consider the process to be the result of the initial screen sinking from y_* down to the sink level which for the time being is at the inner surface of the screen, followed by its rise but, as if from the opposite side of the sink. Since at each of these stages function $a(s)$ is monotonic, hence by virtue of (2.2) function $q(s)$ is also monotonic. In such case under conditions of complete interception by the drain sink of the flow over the screen a specific flow rate Q at the sink corresponds to each value of s in the indicated intervals.

The proposed here physical interpretation of model 2 is also applicable to model 1 under condition that $s < y_A$, since in the latter the depression curve AB must apporach the impermeable boundary $x = 0$ remaining to the right of it. On these physical grounds it is possible to link the two considered models.

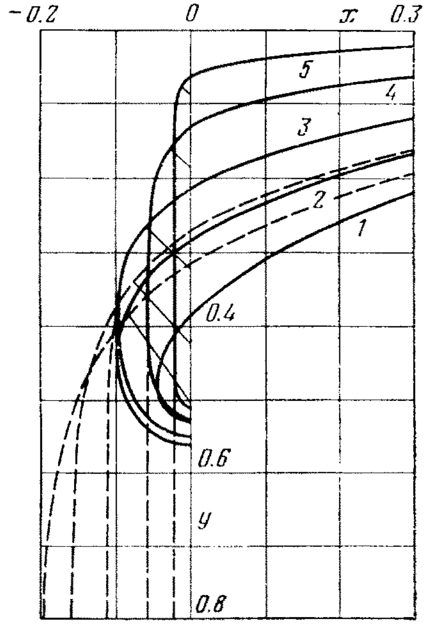

Fig. 3

Indeed, as long as $s < y_*$ and the sink is on the inner surface of the screen, the flow in a particular range of its flow rate for which $s \leqslant y_A \leqslant y_*$, conforms to model 1, and, as the screen is lowered to the sink level followed by its lifting on the other side of the sink, the flow pattern of model 2 obtains.

3. **Model 3.** It will be seen from Fig. 3 that when $s < 0.2$ the depression curve is close to the vertical line $x = -Q$ on the left of the screen over a considerable section. This feature is to some extent implied by the last asymptotic formula of (2.10). A similar pattern should obtain in the case of flow over the screen with subsequent free fall of the stream along it (curve 3 in Fig. 1, a). This model can be obtained as the limit case of model 2 by lowering the sink along the screen external surface to infinity. Taking into account at such passing to limit $\lambda < 1$

and assuming the stream rate of flow to be bounded, for large d in accordance with (1.7) and (1.9) we have

$$\ln\sqrt{1+\alpha^2}\approx\frac{d}{q},\quad a\approx\exp\left(-\frac{2d}{q}\right),\quad f\approx m\sqrt{a};\quad m=\frac{q}{1+Q}\qquad(3.1)$$

As the limit, as $d\to\infty$ from (1.3), (1.4), and (3.1) we obtain

$$\theta=i\frac{1+Q}{2}\int_0^{\zeta}\frac{d\zeta}{\sqrt{\zeta}}=i(1+Q)\sqrt{\zeta},\qquad(3.2)$$
$$z=-Q+(1+Q)\sqrt{\zeta}+iq\,\mathrm{arth}\sqrt{1-\zeta}$$

Then, using (2.6), (2.5), and (3.1) we obtain

$$s=(1+Q)\sqrt{-g}+q\,\mathrm{arsh}\sqrt{-1/g},\quad g=(^1/_2)(1-\sqrt{1+4m^2})\qquad(3.3)$$

It is possible in this case to prove analytically that the stream rate of flow Q monotonically increases, as s is increased, i.e. with the lowering of the screen.

The equation of the depression curve is obtained from (3.2) in the form

$$x=-Q+(1+Q)\,\mathrm{sch}\,(y/q)\qquad(3.4)$$

Model 3 may be considered as the flow of ground water over an underground protuberance whose impermeable contour coincides with one of the streamlines, or may be approximated by the latter as the result of varying the quantity s.

Depression curves of the considered stream calculated by formula (3.4) for the same values of s as in model 2 are shown by dash lines in Fig. 3. The flow rate for each of the values $Q=0.1992,\ 0.1608,\ 0.1130,\ 0.0569$, and 0.0227 were obtained using formulas (3.3) and (3.1). The last three of these flow rates are virtually the same as those obtained earlier for model 2 (for the same s) Q, and the depression curves [for the two models] diverge only in the sink neighborhood. The asymptotic formulas (2.8) and (2.9) for s and y_0 are obtained from (3.1) – (3.3). All this shows the closeness of models 2 and 3 for a particular elevation of the screen, as the result of which the screened sink looses its effect on the structure and seepage properties of the flow. Indeed, for small s and q the depression curve rapidly (almost exponentially in accordance with (3.4)) approaches its asymptote $x=-Q$, and the stream itself immediately after passing over the screen top becomes nearly one-dimensional; in model 2 it is then intercepted by the sink, while in model 3 it continues to move downward.

For high values of Q in conformity with (3.1) –(3.3) we have

$$s\approx1.4153Q,\quad y_0\approx0.9003\sqrt{Q}\qquad(3.5)$$

Formulas (3.5) may also be considered from the point of view of decreases of l with finite values of Q. Expressed in terms of real quantities the second of formulas (3.5) then assumes the form $y_0\approx0.9003\sqrt{Ql}$, and the first remains unchanged, becoming an exact equality at the limit of $l=0$.

The character of dependence between Q (the axis of abscissas), s and y_0 (the right-hand scale on the axis of ordinates), and parameters a and f (the left-hand scale) is illustrated by curves in Fig. 4 calculated for $d = 0.5$ for the three described here models; the curves are denoted by corresponding numerals. In the case of model 1 we assumed that $s = y_A$. In models 2 and 3 y_0 appears with s as an independent parameter of the discharge rate Q. These curves show that already for $s \leqslant 0.25$ functions $y_0(s)$ and $q(s)$ are almost identical for models 2 and 3.

4. **Model 4.** Using the flow defined in model 3 as the background, we assume that a drain sink with a discharge rate Q_d begins to operate on the screen surface, while part of the water is discharged downward at the rate Q_{cb} (Fig. 5, *a*). As Q_d is increased to some specific value $Q_d^{\max}$ dependent on the model geometric parameters l, d and s, the drain intercepts the whole ground stream so that $Q_{cb} = 0$, and either model 2 or 1 is realized (the latter is only possible when the sink is located on the inner screen surface). Thus model 4 is a generalization of the [three] previous ones, which are its limit cases. Function $\theta(\zeta)$ is still represented by formula (1.3) (the position of point F will be discussed later). In region ω (Fig. 5, b) we have now a slit associated with the stream separation. Mapping it onto the half-plane

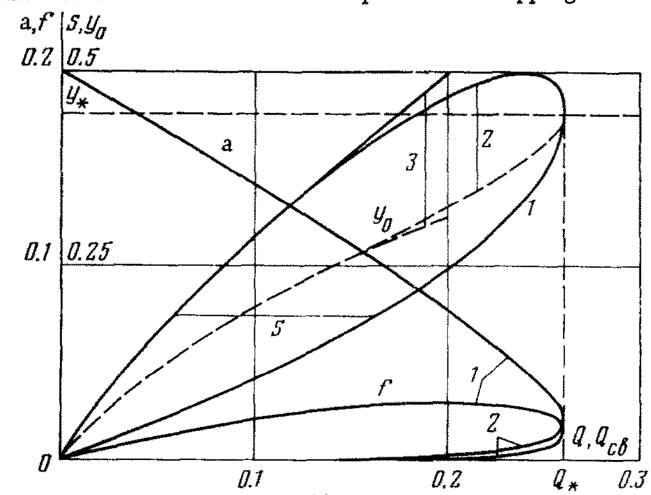

Fig. 4

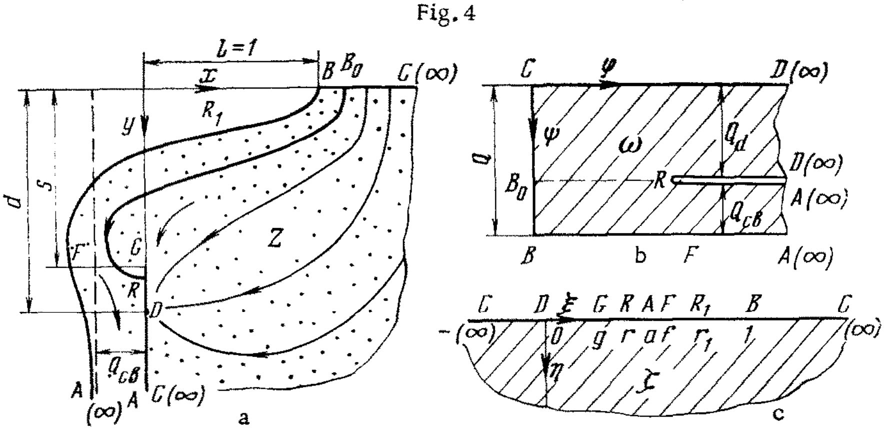

Fig. 5

$\operatorname{Im}\zeta \geqslant 0$ (Fig. 5, c) we obtain

$$\omega = M \int_{\zeta}^{1} \frac{(\zeta - r)\, d\zeta}{\zeta(\zeta - a)\sqrt{1-\zeta}} + iQ = q_d \cdot \operatorname{arth}\sqrt{1-\zeta} + \tag{4.1}$$

$$q_{cb} \operatorname{arth}\sqrt{\frac{1-\zeta}{1-a}} + iQ$$

$$M = \frac{Q_d}{\pi}\frac{a}{r} = \frac{Q_{cb}}{\pi}\frac{a\sqrt{1-a}}{a-r}, \quad q_d = \frac{2}{\pi}Q_d, \quad q_{cb} = \frac{2}{\pi}Q_{cb}$$

$$z = q_d\left(\lambda\sqrt{\frac{\zeta}{a}-1} - \operatorname{arctg}\sqrt{\frac{\zeta}{a}-1}\right) - \tag{4.2}$$

$$Q_{cb} + i\left(q_d \operatorname{arth}\sqrt{1-\zeta} + q_{cb}\operatorname{arth}\sqrt{\frac{1-\zeta}{1-a}}\right)$$

It is interesting to analyze the flow patterns for fixed l, d, and s with varying discharge rate Q_d in the interval $(0, Q_d^{\max})$. For this it is necessary to determine besides l, d, and s, also the discharge rate Q_{cb} and parameters a, f, and g, for which, using (4.1) and (4.2), we obtain the following system of equations (cf. (1.6), (1.7), (2.5), and (2.4)) in dimensionless quantities:

$$q_d\left(\lambda\sqrt{\frac{1}{a}-1} - \operatorname{arctg}\sqrt{\frac{1}{a}-1}\right) = 1 + Q_{cb} \tag{4.3}$$

$$q_d\left(\lambda + \ln\sqrt{\frac{1}{a}}\right) + q_{cb}\operatorname{arth}\sqrt{1-a} = d$$

$$q_d\left(\lambda\sqrt{1-\frac{g}{a}} + \ln\frac{1+\sqrt{1-g}}{\sqrt{a}+\sqrt{a-g}}\right) + q_{cb}\operatorname{arth}\sqrt{\frac{1-a}{1-g}} = s$$

$$f(r-g)\sqrt{a} = r(f-g)\sqrt{(a-g)(1-g)} \quad (r = aq_d/(q_d + q_{cb}\sqrt{1-a}))$$

We restrict ourselves to the preliminary qualitative analysis of the flow. From the last equation, which implies that $w_y(g) = \infty$, we have

$$f - r = f\frac{g-r}{g}\left[1 - \sqrt{\frac{a}{(a-g)(1-g)}}\right] \tag{4.4}$$

If the sink is on the external surface of the screen, i.e. $g < 0$, then, as implied by (1.4), $f > r$. When $Q_d \approx 0$ by virtue of (4.3) we have $r \approx 0$, $f \approx 0$ which means that points R and F lie directly under the sink, and the second of them is the point of maximum pressure in section AD. When $Q_d \approx Q_d^{\max}$, $Q_{cb} = 0$ then in accordance with (4.3) and (4.4)

$$r \approx a, \quad f - a \approx \frac{f}{g}(g-a)\left[1 - \frac{\sqrt{a}}{\sqrt{(a-g)(1-g)}}\right] > 0, \quad \lambda < 1$$

hence with increasing Q_d point F passes onto the depression curve, and at the limit when $Q_{cb} = 0$ we obtain model 2.

When the sink is located on the inner surface of the screen, one of the following variants is possible.

1) $s < y_*(d)$. Since the screen top is then within the range of possible lowering

of the ground water level, the depression curve after reaching some value $Q_d^{\max}$ drops over the sink to the screen top [level] and the flow over the latter ceases. The section AG in region z degenerates into point G which is simultaneously reached by point R ; as the result we have $r = g = a$ in conformity with the last of Eqs. (4.3). If $Q_d < Q_d^{\max}$, when the flow conforms to model 4, we have $0 < g < a$. If then the streams become separated on the inner surface of the screen, i.e. $0 < r < g$ (which occurs at least at small discharge rates Q_d), then by virtue of (4.4) we obtain $0 < f < r$, which means that point F belongs to section RD as the point of maximum pressure on the screen surface in the interval AD. When $Q_d^{\max} < Q_d < Q_d^*$ the flow conforms to model 1.

2) $s = y_*(d)$. With this position of the screen top $Q_d^{\max} = Q_d^*$; total absorption of the ground stream by the drain is attained in the critical mode described in Sect. 1, and for $Q_d < Q_d^{\max}$ the conclusions arrived at in variant 1 are valid.

3) $s > y_*(d)$. In this case, when $Q_d = Q_d^{\max}$ $(Q_{cb} = 0)$, we have model 2, hence $0 < g < a$ throughout the interval $(0, Q_d^{\max})$ of model 4. Then, since $r \approx 0$ for $Q_d \approx 0$, and $r \approx a$ for $Q_{cb} \approx 0$ (see (4.3)), point R passes from the inner to the outer side of the screen, as Q_d is increased. According to (4.4) $\operatorname{sign}(f - r) = \operatorname{sign}(r - g)$ and, consequently, point F is on the same side of the screen as point R bypassing simultaneously with it the screen top and changing from being a stream absorbed by the drain to a free stream; with further increases of Q_d point F passes onto the depression curve.

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